Register number:

Date and session:

ST JOSEPH'S UNIVERSITY, BENGALURU-27
M.SC (MATHEMATICS) - II SEMESTER
SEMESTER EXAMINATION: APRIL, 2023
(Examination conducted in May 2023)

MT 8221: MEASURE AND INTEGRATION
(For current batch students only)

Duration: 2 Hours **Max. Marks:** 50

1. The paper contains **TWO** printed pages and **ONE** part. Attempt any **FIVE FULL** questions.
2. All multiple choice questions have **one or more** correct option. Write **all** the correct options in your answer booklet. True or False questions must be correctly justified.
3. For question 3 answer either parts a and b or parts c and d.
4. Calculators are allowed.

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1. a) Let E_1 and $E_2 \in \mathscr{L}(\mathbb{R}^n)$. Then show that $E_1 \cup E_2 \in \mathscr{L}(\mathbb{R}^n)$. Further, if $E_1 \cap E_2 = \emptyset$ then show that $m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$. **[7]**

   b) True/False: The set $\bigcup_{n=1}^{\infty} \{(x, y) \in \mathbb{R}^2 : y = (n+1)x\}$ has measure zero in $(\mathbb{R}^2, \mathscr{P}(\mathbb{R}^2), \delta_{x_0})$ where x_0 is the origin in $\mathbb{R}^2$. **[3]**

2. a) Let $(X, \mathcal{S}, \mu)$ be a measure space. Show that if $\{E_i\}$ is a countable collection of subsets of X in $\mathcal{S}$ such that $E_1 \supseteq E_2 \supseteq E_3 \cdots$ and $\mu(E_m) < \infty$ for some m then, $\mu\Big(\bigcap_{i=1}^{\infty} E_i\Big) = \lim_{n\to\infty} \mu(E_n)$. **[5]**

   b) Let $(X, \mathcal{S}, \mu)$ be a measure space and let $f : \mathbb{X} \to \mathbb{R}$ be a strictly positive measurable function. Show that the function $\dfrac{1}{f}$ is measurable. **[5]**

3. a) Prove Lusin's Theorem: "Let $E \subset \mathbb{R}^n$ be a set of finite measure. Let $f : E \to \mathbb{R}$ be a measurable function. For every $\varepsilon > 0$ there exists a measurable set $A_\varepsilon \subset E$ such that $m(E \setminus A_\varepsilon) < \varepsilon$ and $f|_{A_\varepsilon}$ is continuous." **[8]**

   b) Let A and B be two subsets of a measure space X. Which of the following is(are) true? **[2]**

i. $A \subset B \implies \chi_A \leq \chi_B$

ii. $\chi_{A \cap B} = \min\{\chi_A, \chi_B\}$

iii. $A \subset B \implies \chi_A \geq \chi_B$

iv. $\chi_{A \cap B} = \max\{\chi_A, \chi_B\}$.

OR

c) Let $(X, \mathcal{S}, \mu)$ be a measure space and let ϕ and ψ be simple functions defined on X. Let $a, b \in \mathbb{R}$. Show that $\int_X (a\phi + b\psi) d\mu = a \int_X \phi d\mu + b \int_X \psi d\mu$. Also show that $\left| \int_X \phi d\mu \right| \leq \int_X |\phi| d\mu$. **[7]**

d) True/False: A constant function on $\mathbb{N}$ is integrable with respect to the Lebesgue measure but not with respect to the counting measure. **[3]**

4. a) Prove the Bounded Convergence Theorem: "Suppose $\{f_n\}$ is a sequence of measurable functions that are all bounded by M and supported on a set E of finite measure and $f_n \to f$ almost everywhere. Then, f is almost everywhere bounded, and supported on E and $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$." **[7]**

b) True/False: The function $f : \mathbb{R} \to [-\infty, \infty]$ defined by $f(x) = \frac{1}{x}$ is integrable on $(\mathbb{R}, \mathcal{P}(\mathcal{R}), \delta_0)$. **[3]**

5. a) Let $(X, \mathcal{S}, \mu)$ be a measure space and let f, g be non-negative measurable functions on X. Let E, F be disjoint measurable subsets of X. Prove that $\int_{E\cup F} f d\mu = \int_E f d\mu + \int_F f d\mu$. Also prove that if $f \leq g$ then $\int_X f d\mu \leq \int_X g d\mu$. **[5]**

b) Let $X = Y = [0, 1]$. Give X the Lebesgue measure m and Y the counting measure μ. Let $f(x, y) = 1$ if $x = y$ and 0 otherwise. Show that $\int_X \int_Y f(x, y) d\mu\, dm \neq \int_Y \int_X f(x, y) dm\, d\mu$. **[5]**

6. a) Prove Minkowski's Inequality: "Let $1 \leq p \leq \infty$. Let f and g be p-integrable. Then $f + g$ is also p-integrable and $\|f + g\|_p \leq \|f\|_p + \|g\|_p$". **[7]**

b) True/False: $L^2(\mathbb{N}) \subseteq L^1(\mathbb{N})$ where the measure considered is the counting measure. **[3]**

7. a) Let $(X, \mathcal{S}, \mu)$ be a measure space and $h \in L^1(X)$ be a non-negative function. For each $E \in \mathcal{S}$, define $\nu(E) = \int_E h d\mu$. Show that ν is a finite measure on $\mathcal{S}$. **[5]**

b) Let $[a, b] \subset \mathbb{R}$ and f be a function of bounded variation on $[a, b]$. Show that f is bounded and $|f|$ is of bounded variation. **[5]**

8. a) Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function such that $f' = 0$ almost everywhere on $[a, b]$. Show that f is a constant. **[7]**

b) Show that a Lipschitz continuous function is of bounded variation. **[3]**