Register number:

Date and session:

# ST. JOSEPH'S UNIVERSITY, BENGALURU-27
## M.SC (MATHEMATICS) - I SEMESTER
## SEMESTER EXAMINATION: OCTOBER, 2023
(Examination conducted in November/December 2023)

## MT 7121: ALGEBRA I
(For current batch students only)

**Duration:** 2 Hours **Max. marks:** 50

---

1. The paper contains **ONE** printed page and ONE part.
2. Attempt any **FIVE FULL** questions.
3. Calculators are allowed.

---

1. a) Let $p$ be prime. Show that a $p$ group has non-trivial center. **[5]**
   b) Given $\beta, \gamma$ in $S_4$ such that $\beta(1) = 4$, $\beta\gamma = (1432)$ and $\gamma\beta = (1234)$ determine $\beta$ and $\gamma$. **[5]**
2. a) Compute the class equation of $S_5$. **[5]**
   b) Let $G$ be a group. Show that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$. **[5]**
3. a) Show that a group of order 12 either has a normal Sylow 3-subgroup or is isomorphic to $A_4$. **[7]**
   b) Show that a group of order 200 is not simple. **[3]**

   **OR**

   c) Let $p$ be a prime. Compute the number of Sylow $p$-subgroups of $GL_2(\mathbb{Z}_p)$. **[6]**
   d) The set $G = \{1, 7, 11, 13, 17, 19, 23, 29\}$ is a group under multiplication modulo 30. Determine the isomorphism class of $G$. **[4]**
4. a) Show that the number of irreducible polynomials of the form $x^2 + ax + b$ in $\mathbb{Z}_p$ is $\dfrac{p(p-1)}{2}$. **[4]**
   b) State Eisenstein's criteria. Show that $f(x) = x^4 + 3x + 3$ is irreducible over $\mathbb{Z}$. **[3]**
   c) Let $n \in \mathbb{Z}$. Show that the polynomial $f(x) = x^3 + nx + 2$ is reducible only for finitely many values of $n$. **[3]**
5. a) Construct a field with 27 elements. **[3]**
   b) Show that $\mathbb{R}[x]/\langle x^2 + 1\rangle$ is isomorphic to $\mathbb{C}$. **[7]**
6. a) Prove that $\mathbb{Z}[x]$ is not a PID. **[7]**
   b) True/False: "Let $R$ and $S$ be rings such that $R \subseteq S$ then, $R$ is a UFD $\implies$ $S$ is a UFD." Justify. **[3]**
7. a) Show that every prime ideal in a PID is a maximal ideal. **[5]**
   b) Show that $\mathbb{Z}[i]$ is an ED. **[5]**