Register number:

Date and session:

ST JOSEPH'S UNIVERSITY, BENGALURU-27
M.SC (MATHEMATICS) - III SEMESTER
SEMESTER EXAMINATION: OCTOBER, 2023
(Examination conducted in November/December 2023)

**MT 9122: Functional Analysis**

**Duration:** 2 Hours **Max. Marks:** 50

1. The paper contains **TWO** printed pages.
2. Attempt any **FIVE FULL** questions.
3. All multiple choice questions have **one or more** correct option(s). Write **all** the correct options.

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1. a) Riesz Lemma states that: "Let $X$ be a normed space. Let $Y$ be a closed proper subspace of $X$ and $Y \neq X$. Let $r$ be a real number such that $0 < r < 1$. Then there exists some $x_r \in X$ such that $||x_r|| = 1$ and $r < dist(x_r, Y) \leq 1$". Give and example to show that the Riesz lemma doesn't hold for $r = 1$. [7]

   b) Which of the following is(are) true? [3]

   I) $c_{00}$ is dense in $c_0$ with $\|\|_\infty$ norm.
   II) $c_{00}$ is dense in $\ell^p$ with $\|\|_p$ norm.
   III) $C^1[0,1]$ is dense in $C[0,1]$ with $\|\|_\infty$ norm.

2. a) Check whether the following linear transformations are bounded linear transformations. If Yes, find it's norm.

   i. Let $X = C^1[0,1]$ with norm $\|x\|_* = \|x\|_\infty + \|x'\|_\infty$ and $Y = C[0,1]$ with the norm $\| \|_\infty$. Let $A : X \to Y$ be the linear transformation defined by $A(x) = x'$. [4]

   ii. Let $X = C^1[0,1]$ and $Y = C[0,1]$ with sup-norm and $T : X \to Y$ be the linear transformation defined by $T(x) = x'$. [2]

   b) Let $(a_{ij})$ be an infinite matrix and $\beta = \sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty$. For an infinite sequence $x = (x_1, x_2, \cdots)$, define a new sequence $A(x)$ with the $i^{th}$ entry as $A(x)_i = \sum_{j=1}^{\infty} a_{ij} x_j$. Show that $A : \ell^\infty \to \ell^\infty$ is a bounded linear operator with norm $\beta$. [4]

3. Show that a real valued normed linear space which satisfies the parallelogram law is induced from an inner product. [10]

**OR**

a) Prove the projection theorem: "Let $H$ be a Hilbert(complete inner product) space and $F$ be a closed subspace of $H$. Then, $H = F \oplus F^{\perp}$ and $(F^{\perp})^{\perp} = F$". [5]

b) Give an example to show that the completeness property is necessary for the conclusion of the Projection theorem. [5]

4. a) Give an example to show that every orthonormal basis of an inner product space need not be a basis. [3]

b) State and prove Riesz-Fischer theorem. [4]

c) Let $X$ be a Hilbert space and $E$ be an orthonormal basis of $X$. Show that $E$ is a basis of $X$ if and only if $X$ is finite dimensional. [3]

5. a) Let $V$ be a normed linear space such that $V^*$ is strictly convex. Show that given a subspace $W$ of $V$ and a continuous linear functional $f$ on $W$, the Hahn-Banach extension of $f$ to all of $V$ is unique. [4]

b) Let $Y$ be a subspace of a normed linear space $X$ and $g \in Y^*$. Show that the set of all Hahn-Banach extensions of $g$ to $X$ is a closed and bounded subset of $X^*$. [3]

c) Which of the following spaces are reflexive. [3]

I) $\mathcal{L}^2[0,1]$ II) $\mathcal{L}^{\infty}[a,b]$ III) $(\mathbb{K}^n, \|\cdot\|_{\infty})$ IV) $(c_{00}, \|\cdot\|_{\infty})$

6. a) Let $C$ be an open and convex subset of a normed linear space $V$ such that $0 \in C$. Define, the Minkowski functional $p$ on $C$. Show that, $p$ satisfies $0 \leq p(x) \leq M\|x\|$ for some $M > 0$ and $p$ is sublinear functional. [6]

b) Let $C$ be a non-empty open convex set in a real normed linear space $V$ and $x_0 \notin C$. Show that there exists $f \in V^*$ such that $f(x) < f(x_0)$ for all $x \in C$. [4]

7. a) State and prove open mapping theorem. [4]

b) Give an example of a continuous linear function $T : V \to W$, where $V$ is not a Banach space, $W$ is a Banach space such that $T$ is not an open map. [6]