Register Number:

Date: 21-11-2020

St. Joseph's College, Autonomous, Bangalore
M.Sc Mathematics-III Semester
End semester Examination: November 2020

MTDE9318: Commutative Algebra

Duration: 2.5 Hours

Max. Marks:70

1. The paper contains two pages.
2. Attempt any **SEVEN FULL** questions.
3. Each question carries 10 marks.

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1. Let A be a ring and let

$$A[[x]] = \{f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \mid a_i \in A\}$$

be the ring of formal power series with coefficients in A.

(a) Show that f is a unit in $A[[x]]$ if and only if a_0 is a unit in A. **[4 marks]**

(b) Show that if f is a nilpotent then a_n is a nilpotent for all $n \geq 0$. Is the converse true? **[6 marks]**

2. (a) Let $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n$ be prime ideals and let I be an ideal contained in $\bigcup_{i=1}^{n} \mathfrak{p}_i$. Show that $I \subseteq \mathfrak{p}_j$ for some $j \in \{1, 2, \cdots, n\}$. **[7 marks]**

(b) Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (n depends on x). Show that every prime ideal in A is maximal. **[3 marks]**

3. (a) Let A be a ring and $\mathrm{Nil}(A)$ its nilradical. Show that the following are equivalent:

   1. A has exactly one prime ideal.
   2. Every element of A is either a unit or a nilpotent.
   3. $A/\mathrm{Nil}(A)$ is a field.

   [5 marks]

(b) Show that $\left(\frac{\mathbb{Q}}{\mathbb{Z}}\right) \otimes_{\mathbb{Z}} \left(\frac{\mathbb{Q}}{\mathbb{Z}}\right) = 0$ **[3 marks]**

(c) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are not isomorphic as $\mathbb{R}$-modules. **[2 marks]**

4. (a) State and Prove the universal property of localization of a ring A at a multiplicative subset S.

[5 marks]

(b) Let A be a ring, let $f : A \to B$ be a homomorphism of rings and let S be a multiplicatively closed subset of A. Let $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.

[5 marks]

5. (a) Let M be an A-module. Show that the following are equivalent:
   1. $M = 0$
   2. $M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ of A.
   3. $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of A.

[5 marks]

(b) Let A be a ring and let M be an A-module. Suppose that $f_1, f_2, \cdots, f_n$ generate the ring A. Prove that if $m \in M$ goes to 0 in each M_{f_i} then $m = 0$.

[5 marks]

6. (a) State and prove Lying Over Theorem. **[6 marks]**

(b) State and prove Going Up Theorem. **[4 marks]**

7. (a) Prove that an A-module is Noetherian if and only if every submodule of M is finitely generated.

[6 marks]

(b) Let M be an A-module and let N_1, N_2 be submodules of M. Show that if M/N_1 and M/N_2 are Noetherian then so is $M/(N_1 \cap N_2)$ **[4 marks]**

8. (a) Let $r(I)$ denote the radical of an ideal I. Show that if $r(I) = \mathfrak{m}$, a maximal ideal then I is a primary ideal. **[5 marks]**

(b) Is it true that "if $r(I) = \mathfrak{p}$, a prime ideal then I is a primary ideal?" Justify. **[5 marks]**

9. (a) Show that in a Noetherian ring A, every ideal contains a power of it's radical. Deduce that, in a Noetherian ring the nilradical is nilpotent. **[5 marks]**

(b) Let A be a Noetherian ring and $f = \sum_{n=0}^{\infty} a_n x^n \in A[[x]]$. Prove that if each a_n is nilpotent then f is nilpotent.

[5 marks]

10. (a) Prove that in an Artin ring every prime ideal is maximal. **[4 marks]**

(b) Prove that a Artin ring has only finitely many maximal ideals. **[6 marks]**

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