Register Number:

Date:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BENGALURU-27
M.SC MATHEMATICS - I SEMESTER
SEMESTER EXAMINATION: OCTOBER, 2021
(Examination conducted in January-March 2022)
MT 7221: REAL ANALYSIS

Duration: 2.5 Hours **Max. Marks:** 70

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1. The paper contains **TWO** pages.
2. Attempt any **SEVEN FULL** questions.
3. All multiple choice questions have **one or more** correct option. Write **all** the correct options in your answer booklet.

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1. a) Give an example of a sequence of partitions of $[0, 1]$. Using sequences of partitions, show that the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = e^x$ is integrable. **[1+6m]**

   b) The value of $\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+4}+\frac{n}{n^2+9}+\frac{n}{n^2+16}+\cdots+\frac{n}{n^2+4n^2}\right)$ is: **[3m]**

   i. $\tan^{-1}(0)$ ii. $\tan^{-1}(1)$ iii. $\tan^{-1}(2)$ iv. $\tan^{-1}(4)$

2. a) Show that if f is integrable on $[a, b]$ and if $f(x) = g(x)$ for all $x \in [a, b]$ except at $\alpha \in [a, b]$, then g is Riemann integrable and $\int_a^b f = \int_a^b g$. **[7m]**

   b) Which of the following is/are true? **[3m]**

   i. If f is continuous and g is integrable then $f \circ g$ is integrable.
   ii. If f and g are integrable then $f \circ g$ is integrable.
   iii. If f is differentiable and g is integrable then $f \circ g$ is integrable.
   iv. If f and g are continuous then $f \circ g$ is integrable.

3. a) If f is continuous on $[a, b]$ and $\int_a^b f = 0$ then prove that there is a point $c \in [a, b]$ such that $f(c) = 0$. Further, if $f \geq 0$ then prove that $f = 0$ for all $x \in [a, b]$. **[7m]**

   b) The value of $\lim_{x\to 0} \frac{1}{x}\int_x^{2x} e^{-t^2}dt$ is, **[3m]**

   i. 1 ii. 0 iii. ∞ iv. oscillates

4. a) Let $\{f_n\}$ be a sequence of functions such that $|f_n(x)| \leq L_n$ for all x where $L_n > 0$ for all n. Show that if f_n converges uniformly to f then f is bounded. **[3m]**

   b) Examine the convergence of $f_n(x) = \frac{x^n}{1+x^n}$ in the range $[0, 2]$. **[4m]**

   c) Let f_n be a sequence that converges to f. In which of the following cases is $\lim_{n\to\infty}\int_0^1 f_n = \int_0^1 f$? **[3m]**

i. $f_n(x) = nx(1-x^2)^n$, $f = 0$

ii. $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $f = 0$

iii. $f_n(x) = \dfrac{x}{n}$, $f = 0$

iv. $f_n(x) = \begin{cases} nx^2, & 0 \leq x \leq 1/n \\ x, & 1/n \leq x \leq 1 \end{cases}$, $f = x$

5. a) Show that the series $\sum_{n=1}^{\infty}(-1)^n \dfrac{x+n}{n^2}$ is uniformly convergent. **[6m]**

b) The power series $\sum_{n=1}^{\infty}(-1)^n \dfrac{(x+1)^n}{n+1}$ converges on **[4m]**

i. $[0,2)$ ii. $(0,2)$ iii. $(-2,0]$ iv. $[-2,0]$

6. a) Show that every superset of an infinite set is infinite and every subset of a finite set is finite **[6m]**

b) Which of the following is/are true for a function $f : X \to Y$? **[4m]**

i. If f is injective and Y is countable then so is X

ii. If f is surjective and Y is countable then so is X

iii. If f is injective and X is countable then so is Y

iv. If f is surjective and X is countable then so is Y.

7. a) Show that $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ is metric on $\mathbb{R}^n$. Further, compute the distance from $(1,1,\cdots,1)$ to $(1,2,3,\cdots,n)$. **[5+2m]**

b) Let $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$. On which of the following sets is d a metric? **[3m]**

i. Set of continuous functions on $[0,1]$

ii. Set of monotonic functions on $[0,1]$

iii. Set of differentiable functions on $[0,1]$

iv. Set of integrable functions $[0,1]$

8. a) Define a closed sphere and a closed set in a metric space. Prove that in a metric space, every closed sphere is a closed set. **[2+4m]**

b) Which of the following are true for subsets A, B of $\mathbb{R}$ with usual metric? **[4m]**

i. $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$

ii. $\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)$

iii. $\overline{(A \cap B)} = \overline{A} \cap \overline{B}$

iv. $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$

9. a) State and prove the Baire's category theorem. **[8m]**

b) Which of the following metric spaces are complete? **[2m]**

i. $(0,1)$ with discrete metric $d(x,y) = 0$ if $x = y$ and 1 otherwise.

ii. $(0,1)$ with metric $d_1(x,y) = \min\{d,1\}$ where d is usual metric

iii. $(0,1)$ with metric $d_1(x,y) = \min\{d,1\}$ where d is discrete metric

iv. $(0,1)$ with usual metric $d(x,y) = |x-y|$

10. a) Prove that a subset of a complete metric space is closed if and only if it is complete. **[7m]**

b) Which of the following is/are true for a continuous function $f : X \to Y$? **[3m]**

i. If $\{x_n\}$ is Cauchy then so is $\{f(x_n)\}$
ii. If $\{f(x_n)\}$ is Cauchy then so is $\{x_n\}$
iii. If $\{x_n\}$ is convergent then so is $\{f(x_n)\}$
iv. If $\{f(x_n)\}$ is convergent then so is $\{x_n\}$